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μ Controllers: Mixed and Fixed

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A method is presented for the synthesis of fixed-order controllers with robustness to mixed real and complex uncertainties. The capabilities of the method are demonstrated on a two-mass/spring benchmark problem and on a flexible satellite example. Several fixed-order mixed μ controllers are designed. A comparison with both full-order and reduced-order controllers indicates that the method is capable of synthesizing low-order controllers achieving robust performance levels similar to full-order designs, and superior to reduced-order controllers.

Introduction

OVER the past decade modern robust control theory has revolutionized multivariable controller design. H_∞ and μ-synthesis techniques consider multiple uncertainty sources at different locations in the plant, external disturbances, as well as performance specifications when designing for robust performance.[1–3] Although these techniques greatly simplify multivariable controller synthesis, they are accompanied by the inherent disadvantage that the resulting compensator is of the same order as the generalized plant. Frequency dependent weights have to be included in the design framework to achieve the desired performance characteristics and to account for the structure in the uncertainty. Thus, the order of the generalized plant is increased resulting in high-order controllers. When implemented, large-order controllers can create time delays, which may be undesirable. One solution to this problem is to use model order reduction on the controller realization. This technique, though, does not consider the properties of the closed-loop system when reducing the order of the controller and, therefore, robustness properties are not guaranteed.

Another approach to avoid controllers of high dimension is to constrain the order of the controller a priori in the design process. The first application of fixed-order dynamic compensation to robust control was developed in Ref. 4 with the introduction of the optimal projection equations. This approach involves the solution of two modified Riccati equations and two modified Lyapunov equations, all coupled through an optimal projection operator. The optimal projection approach is extended in Ref. 5 to a mixed H_2/H_∞ problem where an H_2 overbound on performance is optimized while an H_∞ bound is enforced simultaneously. Both homotopy methods,[6] as well as Newton methods,[7] have been employed to solve the optimal projection equations.

It was shown in Ref. 8 that if either a controller or observer canonical form description is imposed on the compensator structure, the number of free parameters is reduced to its minimal number. Augmenting the compensator to the system, the problem is converted to a static gain output feedback formulation. In Ref. 9, a differential game approach was employed to derive the necessary conditions for such a formulation, and a conjugate gradient algorithm was used for the solution. Recently, homotopy methods have been employed to synthesize fixed-order H_2, H_∞, and mixed H_2/H_∞ controllers.[10]

If multiple uncertainty sources are present in the system, H_∞ controllers are conservative because they do not account for the structure in the uncertainty. The μ-synthesis technique alleviates this problem but is still conservative in that only complex representations of uncertainty models are considered. For real parameter uncertainty this representation can be conservative and may unnecessarily sacrifice robust performance. Therefore, developing methods of design for variations in real parameters is currently an active research area. One of the first approaches to account for both real parameter variations and unstructured uncertainty is given in Ref. 11, where robust stabilizability conditions and a controller synthesis procedure are derived for single input/single output systems. A method to introduce real parameter uncertainty as additional noise inputs and additional weights is presented in Ref. 12. The approach is based on the mixed H_2/H_∞ formulation in Ref. 5 and accounts for the parameter uncertainty via the H_∞ bound. However, the method is conservative because the additional noise inputs and additional weights are not equivalent to the given real parameter uncertainties. The notion of positive realness and a bilinear sector transform are used in Refs. 13 and 14 to approach the problem of real parameter uncertainty. The D-scales of complex μ-synthesis are replaced by the so-called generalized Popov multiplier matrix capturing both real and complex uncertainty effects. Similar to the D–K iteration in μ-synthesis, an iterative controller design scheme is formulated. The authors are currently reformulating their problem as a bilinear matrix inequality feasibility problem for which no global solution presently exists.[15] Another approach using absolute stability theory to introduce real parameter uncertainty is described in Refs. 16 and 17. Based on the mixed H_2/H_∞ framework in Ref. 5, robust controllers are designed by optimizing an H_2 overbound while an analysis test based on a Popov stability multiplier is enforced to account for real uncertainty. Absolute stability theory is used, which allows a characterization of the uncertainties in the system parameters as sector bounded nonlinearities. As in Ref. 5, both full- and fixed-order controllers can be designed with this procedure. The fixed-order design results in additional coupling in the design equations along with an additional equation. Because of the nature of the formulation, the compensator matrices and the Popov multiplier are optimized simultaneously when minimizing the H_2 performance bound. Extensions toward the inclusion of monotonic and odd monotonic nonlinearities to reduce conservatism even further are described in Ref. 16. As a consequence, the resulting stability multipliers are more complex leading to rather intricate expressions for controller synthesis.

A third approach to introduce real parameter uncertainty employs a differential game formulation where uncertain system parameters together with external disturbances and initial states try to maximize a performance index.[18] The resulting controller provides internal stability and robustness toward real parametric uncertainty in a given set. In Ref. 19 the so-called G-scales matrices are introduced in addition to the D-scales to refine the upper bound on the structured singular value μ in case of mixed real/complex uncertainties. A linear matrix inequality approach is chosen in Ref. 20 to compute this upper bound, whereas in Ref. 21 standard optimization techniques are employed in conjunction with branch and bound schemes. More recent results extend the analysis to the synthesis problem by using a D,G–K iteration.[22]

In this paper, a new approach to μ-synthesis is presented by combining a fixed-order controller design technique with the real/complex μ-synthesis procedure from Ref. 22. The resulting μ

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controllers possess the inherent advantages that they are of fixed order and are robust to mixed real/complex uncertainties. The problem formulation is given in Sec. II together with some details on the numerical solution procedure. In Sec. III, some key features of the mixed real/complex μ-synthesis method are presented, and the integration of the fixed-order algorithm is described. A two-mass/spring benchmark problem and a flexible satellite example are investigated in Sec. IV to illustrate the capabilities of the method. Conclusions are presented in Sec. V.

Fixed-Order Controller Design

Problem Formulation

For a standard control problem, the generalized plant is given by

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B_1\boldsymbol{w} + B_2\boldsymbol{u} \tag{1}$$

$$\boldsymbol{z} = C_1\boldsymbol{x} + D_{12}\boldsymbol{u} \tag{2}$$

$$\boldsymbol{y} = C_2\boldsymbol{x} + D_{21}\boldsymbol{w} + D_{22}\boldsymbol{u} \tag{3}$$

where $\boldsymbol{x} \in \mathcal{R}^n$ is the state vector, $\boldsymbol{w} \in \mathcal{R}^{nw}$ is the disturbance vector, $\boldsymbol{u} \in \mathcal{R}^{nu}$ is the control vector, $\boldsymbol{z} \in \mathcal{R}^{nz}$ is the performance vector, and $\boldsymbol{y} \in \mathcal{R}^{ny}$ is the observation vector. It is assumed that (A, B_1, C_1) is stabilizable and detectable, (A, B_2, C_2) is stabilizable and detectable, D_{12} has full column rank, and D_{21} has full row rank. In controller canonical form,[8] the compensator is defined as

$$\dot{\boldsymbol{x}}_c = P^0\boldsymbol{x}_c + N^0\boldsymbol{u}_c - N^0\boldsymbol{y} \tag{4}$$

$$\boldsymbol{u}_c = -P\boldsymbol{x}_c \tag{5}$$

$$\boldsymbol{u} = -H\boldsymbol{x}_c \tag{6}$$

where $\boldsymbol{x}_c \in \mathcal{R}^{nc}$ and $\boldsymbol{u}_c \in \mathcal{R}^{ny}$. P and H are free-parameter matrices, and P^0 and N^0 are fixed matrices of zeros and ones determined by the choice of controllability indices ν_i as follows:

$$P^0 = \text{block diag}\left\{P_1^0, \ldots, P_{ny}^0\right\} \tag{7}$$

$$P_i^0 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}_{\nu_i \times \nu_i} \qquad i = 1, \ldots, ny \tag{8}$$

$$N^0 = \text{block diag}\left\{[0 \quad \cdots \quad 0 \quad 1]^T_{1\times\nu_i}\right\} \qquad i = 1, \ldots, ny \tag{9}$$

The controllability indices ν_i must satisfy the following condition:

$$\sum_i \nu_i = nc \qquad i = 1, \ldots, ny \tag{10}$$

Figure 1 shows the structure of such a controller. With this formulation, the compensator states can be absorbed into the generalized plant. Let

$$\bar{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_c \end{bmatrix} \qquad \bar{\boldsymbol{u}} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{u}_c \end{bmatrix} \tag{11}$$

The augmented system is defined by

$$\begin{aligned} \dot{\bar{\boldsymbol{x}}} &= \begin{bmatrix} A & 0 \\ -N^0C_2 & P^0 \end{bmatrix}\bar{\boldsymbol{x}} + \begin{bmatrix} B_1 \\ -N^0D_{21} \end{bmatrix}\boldsymbol{w} + \begin{bmatrix} B_2 & 0 \\ -N^0D_{22} & N^0 \end{bmatrix}\bar{\boldsymbol{u}} \\ &= \bar{A}\bar{\boldsymbol{x}} + \bar{B}_1\boldsymbol{w} + \bar{B}_2\bar{\boldsymbol{u}} \end{aligned} \tag{12}$$

$$\begin{aligned} \boldsymbol{z} &= [C_1 \quad 0]\bar{\boldsymbol{x}} + [D_{12} \quad 0]\bar{\boldsymbol{u}} \\ &= \bar{C}_1\bar{\boldsymbol{x}} + \bar{D}_{12}\bar{\boldsymbol{u}} \end{aligned} \tag{13}$$

$$\bar{\boldsymbol{y}} = [0 \quad I]\bar{\boldsymbol{x}} = \bar{C}_2\bar{\boldsymbol{x}} \tag{14}$$

$$\bar{\boldsymbol{u}} = -\begin{bmatrix} H \\ P \end{bmatrix}\bar{\boldsymbol{y}} = -G\bar{\boldsymbol{y}} \tag{15}$$

Equations (12–15) define a static gain output feedback problem where the compensator is represented by a minimal number of free parameters in the design matrix, G. The closed-loop system is given by

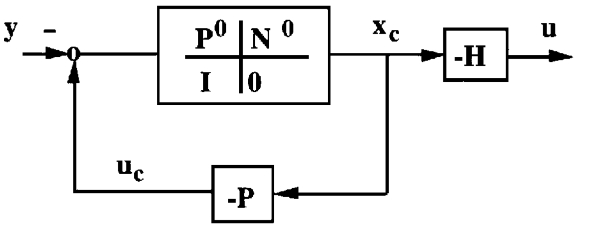

Fig. 1 Compensator in controller canonical form.

$$\begin{aligned} \dot{\bar{\boldsymbol{x}}} &= (\bar{A} - \bar{B}_2G\bar{C}_2)\bar{\boldsymbol{x}} + \bar{B}_1\boldsymbol{w} \\ &= \tilde{A}\bar{\boldsymbol{x}} + \tilde{B}\boldsymbol{w} \end{aligned} \tag{16}$$

$$\boldsymbol{z} = (\bar{C}_1 - \bar{D}_{12}G\bar{C}_2)\bar{\boldsymbol{x}} = \tilde{C}\bar{\boldsymbol{x}} \tag{17}$$

In the H_∞ problem, the objective is to minimize an overbound γ on the ∞-norm of the closed-loop transfer function from disturbance inputs $\boldsymbol{w}$ to performance outputs $\boldsymbol{z}$ given by

$$T_{zw} = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} \tag{18}$$

In Ref. 9, a differential game formulation together with a worst-case disturbance model was used to derive the fixed-order H_∞ optimization problem

$$\min_{G \in G_\gamma} \left\{J_\gamma(G) = \text{tr}\left\{Q_\infty\tilde{B}\tilde{B}^T\right\}\right\} \tag{19}$$

where $G_\gamma = \{G \in R^{(nu+ny)\times nc} : \tilde{A}$ is stable and $\|T_{zw}\|_\infty < \gamma\}$ and Q_∞ is the positive semidefinite solution of

$$\tilde{A}^TQ_\infty + Q_\infty\tilde{A} + \tilde{C}^T\tilde{C} + \gamma^{-2}Q_\infty\tilde{B}\tilde{B}^TQ_\infty = 0 \tag{20}$$

To obtain the H_∞-optimal compensator, the Lagrangian is defined as

$$\begin{aligned} \mathcal{L}(Q_\infty, L, G) = \text{tr}\Big\{&Q_\infty\tilde{B}\tilde{B}^T + \big(\tilde{A}^TQ_\infty + Q_\infty\tilde{A} + \tilde{C}^T\tilde{C} \\ &+ \gamma^{-2}Q_\infty\tilde{B}\tilde{B}^TQ_\infty\big)L\Big\} \end{aligned} \tag{21}$$

where L represents the Lagrangian multiplier. Matrix gradients are taken to determine the first-order necessary conditions for an H_∞-optimal fixed-order controller

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial Q_\infty} &= \big(\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^TQ_\infty\big)L \\ &\quad + L\big(\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^TQ_\infty\big)^T + \tilde{B}\tilde{B}^T = 0 \end{aligned} \tag{22}$$

$$\frac{\partial\mathcal{L}}{\partial L} = \tilde{A}^TQ_\infty + Q_\infty\tilde{A} + \tilde{C}^T\tilde{C} + \gamma^{-2}Q_\infty\tilde{B}\tilde{B}^TQ_\infty = 0 \tag{23}$$

$$\frac{\partial\mathcal{L}}{\partial G} = 2\big(\bar{D}_{12}^T\bar{D}_{12}G\bar{C}_2 - \bar{D}_{12}^T\bar{C}_1 - \bar{B}_2Q_\infty\big)L\bar{C}_2^T = 0 \tag{24}$$

A similar approach has been used in Ref. 10 to derive the necessary conditions for the H_2 and the mixed H_2/H_∞ problem, also for fixed-order controllers.

A conjugate gradient method was used in Ref. 9 to solve the fixed-order H_∞ problem. However, a shortcoming of conjugate gradient algorithms is that convergence slows down near the optimum. Also, a starting guess is required for the compensator gain matrix, which stabilizes the closed-loop system. In this study, a homotopy method is employed to design fixed-order H_∞ controllers. This method greatly simplifies the problem of finding an initial stabilizing controller. Additionally, the prediction/correction scheme used in a homotopy algorithm allows for a naturally automated procedure to synthesize fixed-order controllers.

Homotopy Algorithm

A homotopy method is employed to calculate the fixed-order controller design. For details regarding the use of homotopy methods, the reader should consult Refs. 6 and 23. The particular equations arising from the implementation of the fixed-order formulation in the homotopy framework are provided in Ref. 10.

Homotopy methods begin with a simplified version of the original problem, which has a known or easily calculated solution. This starting guess is then gradually deformed until the solution of the original problem is obtained. In the homotopy algorithm employed, an initial guess for the compensator gain matrix G_0 is obtained by a full-order, low-authority H_2 design. It has been observed numerically that order reduction techniques, although not always reliable, work best for low-authority linear quadratic Gaussian controllers.[6] However, if the open-loop system is unstable, order reduction techniques are not reliable even for low-authority H_2 designs. Thus, it may be required to shift individual poles of the system into the left-half plane to obtain a reducible H_2 controller. This controller is then reduced to the desired order using a balanced order reduction scheme[24] and subsequently transformed into controller canonical form.[25]

The philosophy of imposing a canonical form on the compensator structure minimizes the number of free parameters but can also lead to numerical ill-conditioning of the problem.[6] A balancing transformation,[24] which does not affect the controller characteristics, relaxes the strict structure in the P^0 and N^0 matrices in Eqs. (7–9) and improves the conditioning of the problem considerably.

In the H_2 problem, the homotopy transforms the gain G_0 of the low-control authority H_2 design for a stable system to the gain G_2 of a full-authority H_2 design for a possibly unstable system. This is done by gradually deforming the weights on control cost and measurement noise, as well as the system A matrix, while proceeding along the homotopy path until the desired problem formulation is recovered.

Because the same procedure to find an initial guess is employed in the H_∞ problem, an H_2 homotopy has to be completed first to restore the original problem. Then, a second homotopy is appended to perform the H_∞ design. The upper bound for $\|T_{zw}\|_\infty$, γ, is reduced from an initial value (greater than the ∞-norm of the H_2 design) toward its minimal value for which a controller exists such that $\|T_{zw}\|_\infty < \gamma$. The entire procedure is illustrated in Fig. 2. For an H_2 design, the procedure is stopped after the first homotopy path $G_0 \to G_2$, whereas a second homotopy $G_2 \to G_\infty$ reduces γ in an H_∞ design. In principle, it is possible to use a direct homotopy from G_0 to G_∞ while simultaneously deforming the weights and γ. However, the transition low-to-full authority and/or stable-to-unstable open-loop system always has to be completed to obtain the desired plant, whereas the minimum value of γ that can be achieved is usually not known beforehand. Thus, the proposed two-step procedure is more feasible for practical purposes.

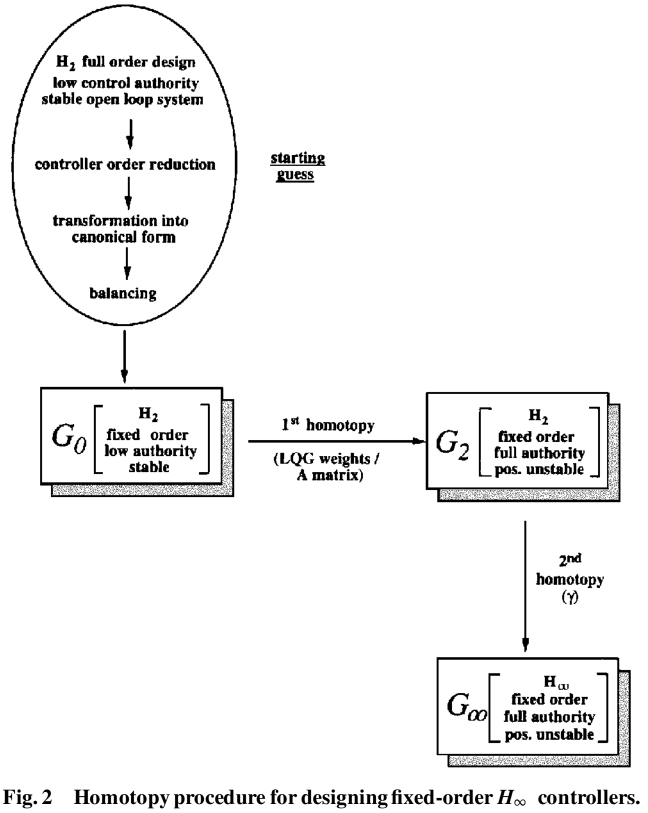

Fig. 2 Homotopy procedure for designing fixed-order H_∞ controllers.

Homotopy algorithms are rather sensitive to numerical ill-conditioning. If the plant and the adjoined weighting functions are not well defined or of large order, the Hessian matrix can exhibit significant ill-conditioning. Because the inverse of the Hessian is used in both the prediction and the correction step of the homotopy procedure, it is of importance to define a reasonable system. The size of the Hessian matrix is determined by the number of control inputs nu, the number of measurements ny, and the number of controller states nc. The relationship

$$n_{H_\theta} = [(nu + ny) \cdot nc]^2 \tag{25}$$

defines the number of elements in the Hessian matrix. This illustrates that for large systems with many control and measurement signals as well as for large controllers the possibility of an ill-conditioned Hessian arises. Nevertheless, a successful homotopy requires robustness to nearly singular and/or indefinite Hessian matrices. Methods to robustify the Hessian matrix and to improve the conditioning of the algorithm are described in detail in Ref. 26.

New Approach to μ-Synthesis

Mixed Real/Complex μ-Synthesis

An in depth discussion of mixed real/complex μ-analysis and synthesis using D,G–K iteration is presented in Refs. 19, 21, and 22. Hence, only some key features are reviewed.

Consider the robust performance problem depicted in Fig. 3. The definition of the structured singular value μ depends on the underlying block structure of the uncertainties. The following types of uncertainties are considered: possibly repeated real scalars δ^r, possibly repeated complex scalars δ^c, and full complex blocks Δ^C. In μ-synthesis, it is desired to design a controller K achieving

$$\inf_{K \in \mathcal{K}_S} \sup_{\omega \in \mathcal{R}} \mu[M(P, K)(j\omega)] \tag{26}$$

where μ is the structured singular value, $M(P, K)$ is the linear fractional transformation of the plant P and the controller K, and $\mathcal{K}_S$ is the set of all real-rational, proper controllers that nominally stabilize P. Unfortunately, the structured singular value cannot be calculated directly. To formulate an upper bound on $\mu(M)$, two sets of block diagonal frequency dependent scaling matrices $\mathcal{D}$ and $\mathcal{G}$ are defined. The upper bound on μ is then given as[19]

$$\mu(M) \le \inf_{\substack{D \in \mathcal{D} \\ G \in \mathcal{G}}} \inf_{\substack{\beta \in \mathcal{R} \\ \beta > 0}} \left\{ \beta : \bar{\sigma}\left[\left(\frac{DMD^{-1}}{\beta} - jG\right) \times (I + G^2)^{-\frac{1}{2}}\right] \le 1 \right\} \tag{27}$$

where $\bar{\sigma}(\cdot)$ denotes the maximum singular value. The μ-analysis problem in Eq. (27) can be combined with Eq. (26) to form the

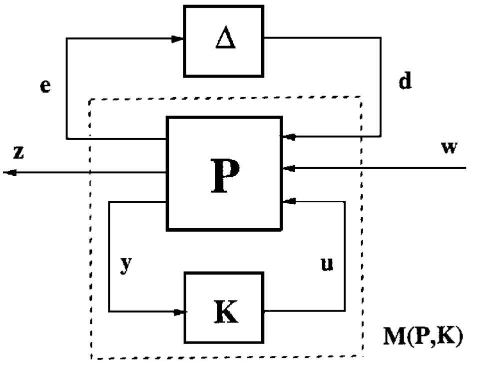

Fig. 3 Framework for robust control design.

mixed μ-synthesis problem

$$\inf_{K\in\mathcal{K}_S}\ \sup_{\omega\in\mathcal{R}}\ \inf_{\substack{D(\omega)\in\mathcal{D}\\ G(\omega)\in\mathcal{G}}}\ \inf_{\substack{\beta(\omega)\in\mathcal{R}\\ \beta(\omega)>0}} \{\beta(\omega):\ \Gamma\le 1\} \tag{28}$$

where

$$\Gamma = \bar{\sigma}\left\{\left[\frac{D(\omega)M(P,K)D^{-1}(\omega)}{\beta(\omega)} - jG(\omega)\right][I+G^2(\omega)]^{-\frac{1}{2}}\right\} \tag{29}$$

As in complex μ-synthesis, the problem of finding optimal $D(\omega)$, $G(\omega)$, and $\beta(\omega)$ for a given controller K is quasiconvex. Real-rational transfer matrices $D(s)$ and $G(s)$ are fitted to $D(\omega)$ and $jG(\omega)$ and augmented to the plant in such a way that the resulting interconnection is stable. This involves state-space factorizations as described in Ref. 22. Holding D, G, and β fixed, the problem of finding K is reduced to a standard H_∞ problem, which is convex in the full-order case. The overall procedure, however, is not convex. This leads to an iterative approach denoted D,G–K iteration[22] where alternately optimal full-order H_∞ controllers for constant scaling matrices and optimal scalings for a constant controller are designed until achievable robust performance cannot be improved anymore.

Designing Mixed μ Controllers of Fixed Order

As shown in the preceding section, H_∞ controller design is a subproblem in mixed μ-synthesis using D,G–K iteration. The augmentation of the transfer matrices $D(s)$ and $G(s)$ to the generalized plant increases the dimension of the problem and, subsequently, the dimension of the full-order controller. By replacing the full-order H_∞ controller design in the D,G–K iteration procedure with the fixed-order technique described earlier, μ controllers of significantly lower order can be synthesized.

A critical step in the fixed-order design technique is obtaining the initial guess for the fixed-order compensator to start the homotopy procedure. Because a full-order mixed μ controller is usually designed first to determine the lowest possible upper bound on mixed μ that can be achieved, an attempt can be made to reduce this full-order design to the desired order using an order reduction scheme. If this reduced-order controller stabilizes the closed-loop system, it is used as a starting guess for the fixed-order algorithm. Because the reduced-order compensator already represents an H_∞ design, the homotopy procedure is entered at the position of G_2 in Fig. 2, and the algorithm is used to improve over the reduced-order controller. However, the D- and G-scales have been optimized for a full-order design and may not be representative for a lower-order controller. This may result in a possibly conservative design, and additional D,G–K iterations with fixed-order controllers may need to be performed to obtain the suboptimal design. In this case, the fixed-order compensator is used to compute new D- and G-scales and serves as a starting guess for the homotopy algorithm for a new fixed-order controller synthesis using the improved scaling matrices. These fixed-order D,G–K iterations are repeated until there is no further improvement.

If order reduction of the full-order mixed μ controller results in a compensator that is not internally stabilizing, the entire fixed-order design procedure beginning with an H_2 compensator as shown in Fig. 2 needs to be performed. The H_2 design can be obtained either for the augmented system from the full-order design, which contains D- and G-scales, or for the original generalized plant. In the first iteration, the homotopy algorithm starts with the H_2 controller as initial guess. In subsequent D,G–K iterations, the previous fixed-order controller is used to start the code.

An alternative to starting with an H_2 controller (when order reduction of the μ design to the desired order fails) is to reduce the μ controller to a higher order for which the reduced controller is internally stabilizing. As an intermediate result, this reduced μ controller serves as a starting guess for a fixed-order design as described earlier. Once the fixed-order design is optimized, an attempt can be made to further reduce this fixed-order controller to the desired order or, if necessary, to another intermediate higher-order design. Similarly, it can happen that order reduction succeeds in providing an internally stabilizing controller that is of lower order than desired. This controller can also serve as a starting guess for a higher fixed-order design. Additional states can be augmented to achieve the desired controller order while a zero block at the corresponding location in the output matrix maintains the transfer function characteristics of the initial stabilizing lower-order controller.

Another possibility to obtain a low-order starting guess is to design a μ controller with constant D-scales. This provides initial information about the uncertainty structure without increasing the number of states over a full-order H_∞ design.

In the ideal case it is expected that all approaches achieve the same level of robust performance, no matter which starting guess procedure for the fixed-order D,G–K iteration is used. However, since fixed-order controller design has not been proven to be convex, different initial guesses may result in different local optima. Also, numerical difficulties associated with the homotopy algorithm may prevent a design that is close to the optimum, especially when multiple fixed-order D,G–K iterations are performed. Thus, starting with a reduced-order mixed μ design is preferred because it has been observed that only a few additional iterations are required to obtain a suboptimal fixed-order μ controller. In case of an H_2 design based starting guess, using the augmented plant can have advantages because less D,G–K iterations are required than when starting with the generalized plant and no a priori available information on the uncertainties.

Design Examples

Benchmark Problem

A simple yet meaningful benchmark problem for robust control design was introduced in Ref. 27. Consider a two-mass/spring system, which is a generic model of an uncertain dynamic system with noncollocated sensor and actuator. The system dynamics are given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -k/m_1 & k/m_1 & 0 & 0\\ k/m_2 & -k/m_2 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 1/m_2\end{bmatrix} w + \begin{bmatrix}0\\ 0\\ 1/m_1\\ 0\end{bmatrix} u \tag{30}$$

$$z = x_2 \tag{31}$$

$$y = x_2 + v \tag{32}$$

where x_1 and x_2 are the positions of body 1 and body 2, respectively; x_3 and x_4 are the velocities of body 1 and body 2, respectively; w is the disturbance acting on body 2; u is the control input acting on body 1; z is the output to be controlled; y is the measurement; and v is the sensor noise. The masses of the bodies are m_1 and m_2, and the spring constant is k. Their nominal values are $m_1 = m_2 = 1$ and $k = 1$. The design specifications in Ref. 27 require that for a unit impulse disturbance exerted on body 2, the controlled output has a settling time of about 15 s for the nominal system, and that the closed-loop system be stable for $0.5 \le k \le 2.0$ and $m_1 = m_2 = 1$. Additionally, the closed-loop system should be insensitive to high-frequency sensor noise and reasonable control effort should be used.

A complex μ-synthesis design for this problem was performed in Ref. 28. The weights chosen in Ref. 28 are also selected in this study. They are

$$W_w = 1, \qquad W_v = 0.01$$
$$W_u = 15, \qquad W_z = 0.06 \tag{33}$$

where W_w is the weight on the disturbance w, W_v is the weight on the sensor noise v, W_u is the weight on the control u, and W_z is the weight on the performance variable z. The uncertainty in the spring stiffness k is represented by a scalar perturbation to the nominal model. This perturbation can be rearranged to obtain input and

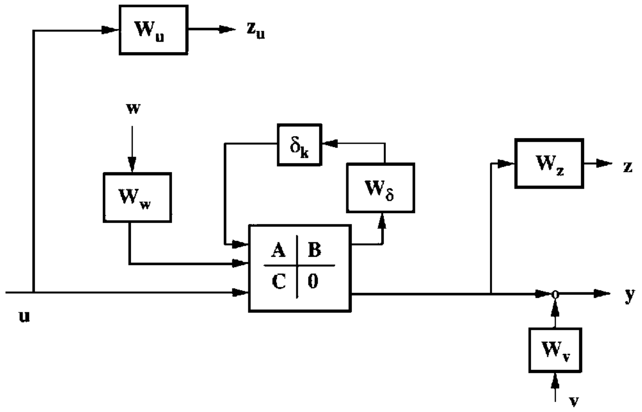

Fig. 4 Generalized plant for benchmark problem.

output quantities d_Δ and e_Δ. The matrices B_Δ, C_Δ, and D_Δ can be chosen from the uncertain model

$$\dot{x} = Ax + Bu + B_\Delta d_\Delta \qquad e_\Delta = C_\Delta x + D_\Delta u$$
$$y = Cx + Du \qquad d_\Delta = \Delta\, e_\Delta \tag{34}$$

If a scaling has been applied so that the values of Δ range between -1 and 1, Eq. (34) represents the set of all possible models. For the case of uncertainty in k, the uncertainty matrices are

$$B_\Delta = [0 \quad 0 \quad -1/m_1 \quad 1/m_2]^T \tag{35}$$

$$C_\Delta = [1 \quad -1 \quad 0 \quad 0] \tag{36}$$

$$D_\Delta = 0 \tag{37}$$

$$\Delta = \delta_k \tag{38}$$

To achieve $0.5 \le k \le 2.0$, the nominal value of k for robust control design is selected to be $k_0 = 1.25$, and the uncertainty weight is

$$W_\delta = 0.75 \tag{39}$$

which normalizes the uncertainty, i.e., $\delta_k \in [-1 \ 1]$. The generalized plant is illustrated in Fig. 4. A characteristic feature of the design in Ref. 28 is that the performance block was chosen to be a diagonal matrix with two independent 1×1 blocks. This decouples the performance specifications that u be small and that z respond quickly to a unit impulse in w. The generalized plant remains the same for both the complex and the mixed μ designs. The additional information on the real parameter uncertainty is captured exclusively by the G-scales matrices of the mixed μ-synthesis procedure. Although only real parameter uncertainty is present in this example, it still constitutes a mixed μ problem: augmenting the complex performance block to the uncertainty in the robust performance problem recovers the mixed μ formulation.

In a first design step, both full-order complex and mixed μ controllers were synthesized. The 12th-order complex μ design achieved a peak value of the upper μ bound of $\mu = 1.321$. Additional states due to the G-scaling matrix resulted in a 16th-order mixed μ controller, which could improve over the complex one, and achieved a peak value of 0.891 for the mixed upper μ bound. To illustrate the benefit of the mixed μ approach, the time responses of the closed-loop systems with the full-order controllers for a unit impulse disturbance on body 2 are shown in Fig. 5. The initial deflection of body 2 is considerably smaller when the mixed μ controller is implemented. Moreover, the responses are similar for all values of the spring stiffness k when the mixed μ controller is used, whereas the complex design results in significant oscillations for $k = 0.5$. Neither controller is quite able to satisfy the 15-s settling time requirement, which is somewhat inconsistent with the results in Ref. 28. However, because certain specific details of the design procedure could not be compared with Ref. 28 (e.g., order of the D-scales used, tolerance in bisection steps in H_∞ design, etc.), this discrepancy is acceptable. The control responses for the nominal

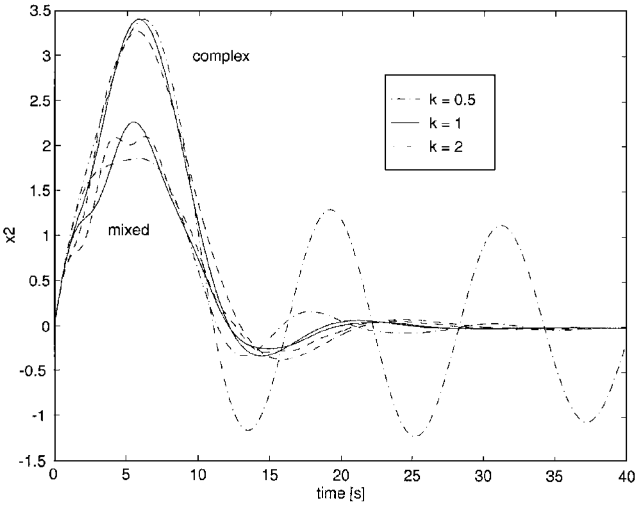

Fig. 5 Position of body 2, full-order controllers and varying spring stiffness *k*.

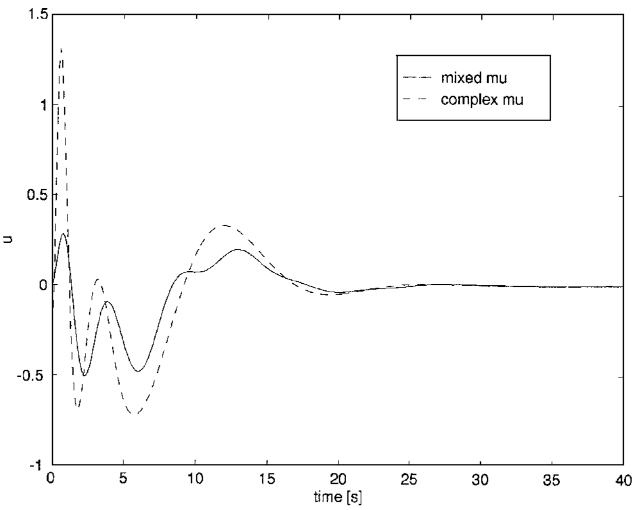

Fig. 6 Control effort of full-order controllers for nominal system, *k* = 1.

system ($k = 1$) for both controllers are shown in Fig. 6. The control effort is significantly reduced for the mixed μ design.

Employing a balanced model reduction scheme, the mixed μ controller could be reduced to third order. This increased the peak value of the upper μ bound to $\mu = 3.622$. Using this reduced μ controller as a starting guess for the fixed-order algorithm described in the preceding section, a fixed third-order mixed μ controller could be designed achieving a peak value of $\mu = 0.914$. Only one additional fixed-order D,G–K iteration was necessary to recover the performance level of the full-order controller. The upper μ bounds are illustrated in Fig. 7. The improvement of both full- and fixed-order mixed μ designs over the complex μ controller is clearly visible. Figure 8 shows the time responses for the same unit impulse acting on body 2, now with the third-order mixed μ controllers implemented. The reduced-order controller leads to considerable oscillations for all values of k and exhibits the trend to a steady-state error. The fixed-order controller, however, reproduces the full-order response from Fig. 5. Accordingly, the control response for the fixed-order controller is similar to the full-order mixed μ response in Fig. 6, whereas the control response of the reduced third-order design exhibits stronger oscillations (not shown).

All μ controllers guarantee robust stability for an allowable variation in the spring stiffness of $0.5 \le k \le 2.0$. However, the actual range of k for which the closed-loop system is stable will usually be larger. The root locus plot in Fig. 9 of the closed-loop system poles for a varying spring stiffness k when the fixed third-order mixed μ controller is implemented reveals that the system is actually stable for $0.36 \le k \le 4.32$. This constitutes a significantly

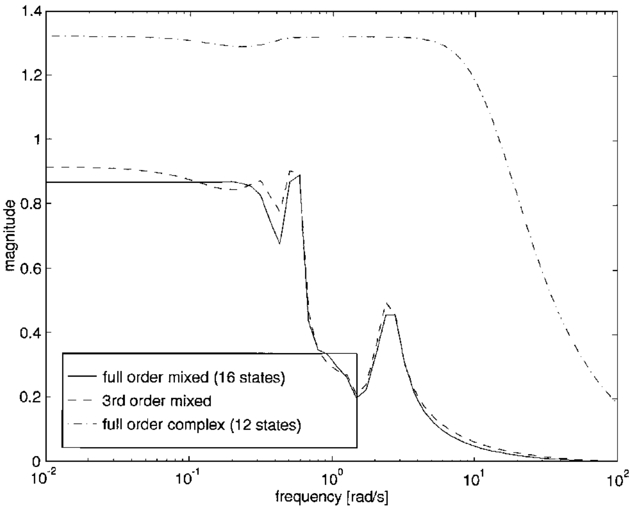

Fig. 7 Upper μ bounds for benchmark problem.

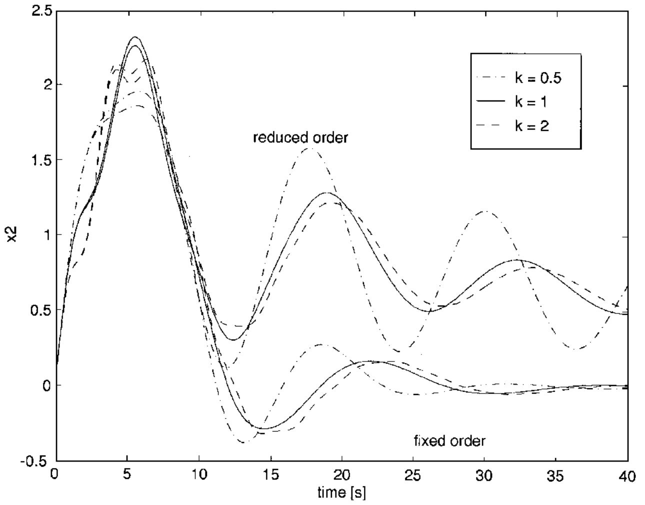

Fig. 8 Position of body 2, third-order mixed μ controllers and varying spring stiffness k.

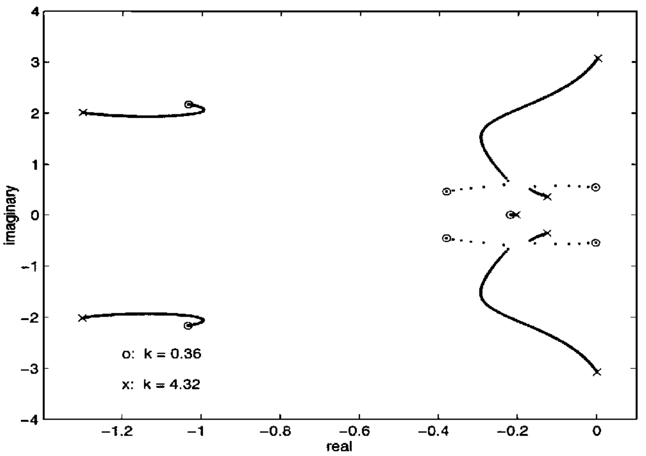

Fig. 9 Parameter root locus for spring stiffness k, third-order mixed μ design.

larger range of k for which the system can be stabilized with the designed controller than originally required. The closeness of actual and guaranteed stability bounds for small values of k indicates that a reduction in k is harder to achieve without loosing closed-loop stability than an increase. A comparison with the reduced fourth-order complex μ controller in Ref. 28 shows that this controller stabilized the system for $0.46 \leq k \leq 6.8$. In Ref. 17, a Popov controller considering real parameter uncertainty in k was designed for the same system. This controller was also fourth-order and achieved closed-loop stability for $0.45 \leq k \leq 2.05$. This shows that the fixed third-order mixed μ controller compares well with previous designs and is able to improve upon the more critical lower stability bound for k.

The benchmark problem illustrates two significant benefits of the controller design procedure: 1) the mixed μ-synthesis method is able to improve over complex μ-synthesis and 2) the fixed-order mixed μ controller improves over a corresponding reduced-order design. The resulting fixed-order mixed μ controller recovers full-order robust performance levels while considerably reducing controller complexity from 16 to 3 states.

Flexible Satellite Example

The second example is a robust attitude control system design for a flexible spacecraft, which is taken from Ref. 29. In this problem, the actuators and sensors are collocated. Focusing on the yaw axis only, the model dynamics are

$$\mathcal{M}\ddot{z} + \mathcal{D}\dot{z} + \mathcal{K}z = \mathcal{B}u \tag{40}$$

where $z = [z_1 \;\; z_2]^T$ and z_1 is the yaw angle displacement in radians, z_2 is the modal displacement used to represent the flexible dynamics, and u is the control torque in inch-pounds. The coefficient matrices are

$$\mathcal{M} = \begin{bmatrix} 77511 & 248.1 \\ 248.1 & 1 \end{bmatrix} \qquad \mathcal{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0.002288 \end{bmatrix}$$
$$\mathcal{K} = \begin{bmatrix} 0 & 0 \\ 0 & 0.098696 \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{41}$$

Augmenting the model by the integral of z_1 and rewriting it in first-order form leads to the system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \tag{42}$$

where

$$\boldsymbol{x} = \left[z^T \;\; \dot{z}^T \;\; \int z_1 \; \mathrm{d}t \right]^T$$

and

$$A = \begin{bmatrix} 0_{2\times 2} & I_2 & 0_{2\times 1} \\ -\mathcal{M}^{-1}\mathcal{K} & -\mathcal{M}^{-1}\mathcal{D} & 0_{2\times 1} \\ [1 \;\; 0] & 0_{1\times 2} & 0 \end{bmatrix} \tag{43}$$

$$B = \begin{bmatrix} 0_{2\times 1} \\ \mathcal{M}^{-1}\mathcal{B} \\ 0 \end{bmatrix} \tag{44}$$

The measured variables are

$$\boldsymbol{y} = \left[z_1 \;\; \dot{z}_1 \;\; \int z_1 \; \mathrm{d}t \right]^T$$

The purpose of the attitude control system is to maintain satellite pointing accuracy during a station keeping maneuver. The system specifications considered in this design are 1) to maintain a 0.02 deg (3.5×10^{-4} rad) pointing accuracy for a 1.0-in.-lbf step disturbance torque at the plant input and 2) to maintain this pointing accuracy in the presence of a 25% variation in the structural frequency due to uncertainty in the stiffness matrix.

It was shown in Ref. 29 that for this example a 25% variation in the structural frequency is represented by a stiffness variation of $0.053296 \leq k \leq 0.154953$, where k is the only nonzero element in $\mathcal{K}$. Therefore, the nominal value of k for robust control design was selected to be $k_0 = 0.104124$ and the uncertainty weight is

$$W_\delta = 0.050829 \tag{45}$$

According to the framework in Eq. (34), the uncertainty matrices are

$$B_\Delta = [0 \quad 0 \quad m_{12} \quad m_{22} \quad 0]^T \tag{46}$$

$$C_\Delta = [0 \quad 1 \quad 0 \quad 0 \quad 0] \tag{47}$$

$$D_\Delta = 0 \tag{48}$$

where m_{12} and m_{22} are the (1, 2) and (2, 2) elements of $-\mathcal{M}^{-1}$, respectively. The weights reflecting the performance specifications are selected to be

$$W_T = 1 \tag{49}$$

$$W_z = \operatorname{diag}[400 \quad 1 \quad 20] \tag{50}$$

$$W_u = 0.1 \tag{51}$$

$$W_v = 10^{-6} \cdot I_3 \tag{52}$$

where W_T is the weight on the torque disturbance at the plant input, W_z is the performance weight on the measured variables y, W_u is the weight on the control u, and W_v is the sensor noise on the measured variables y.

As in the previous example, full-order complex and mixed μ controllers were designed first. The peak value of the (scaled) upper μ bound of the complex design was $\mu = 0.923$. The mixed μ controller could again improve over this result and achieved $\mu = 0.768$. Controller complexity, however, increased from 11 states of the complex controller to 17 states of the mixed design. Using various order reduction schemes, the lowest order for which a reduced mixed μ controller was internally stabilizing was seven. The upper μ value of the reduced-order design was $\mu = 1.091$, indicating a loss in robust performance. Using the reduced controller as a starting guess for the fixed-order design, full-order performance could be fully recovered and the fixed seventh-order controller also achieved $\mu = 0.768$. Because even this seventh-order controller could not be reduced any further without loosing internal stability, an alternative was to design a complex μ controller with constant D-scales. This resulted in a fifth-order complex μ controller achieving $\mu = 0.945$. Combining this controller with the augmented generalized plant from the full-order mixed μ design containing D- and G-scales as a starting guess for the fixed-order procedure, a fixed fifth-order mixed μ controller was obtained that could also recover the robust performance level of the full-order mixed μ design by achieving $\mu = 0.776$. The μ values of all designs are summarized in Table 1. Figure 10 illustrates the the upper μ bounds of the closed-loop systems with the full- and fixed-order controllers implemented. The improvement of the mixed over the complex μ designs is obvious, and it can be observed how the fixed-order controllers recover the full-order performance.

Table 1 Peak values of upper μ bounds for full-, reduced-, and fixed-order μ controllers, satellite example

	Complex μ, full (number of states)	Mixed μ, full (number of states)	Mixed μ, reduced	Mixed μ, fixed
Full order	0.923 (11)	0.768 (17)	——	——
Seventh order	——	——	1.091	0.768
Fifth order	0.945	——	unstable	0.776

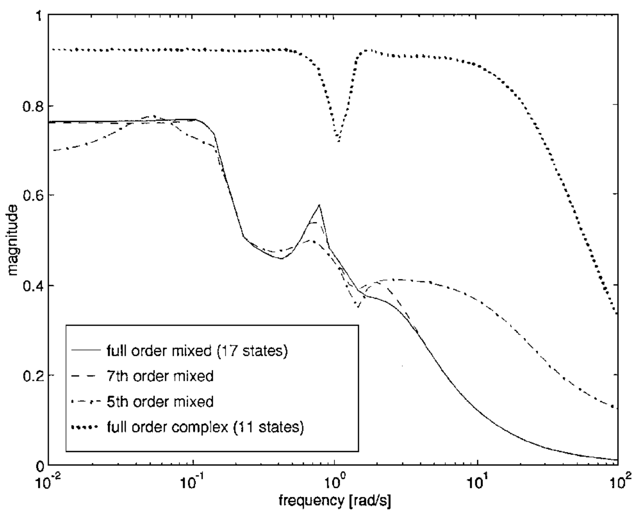

Fig. 10 Upper μ bounds for flexible satellite example.

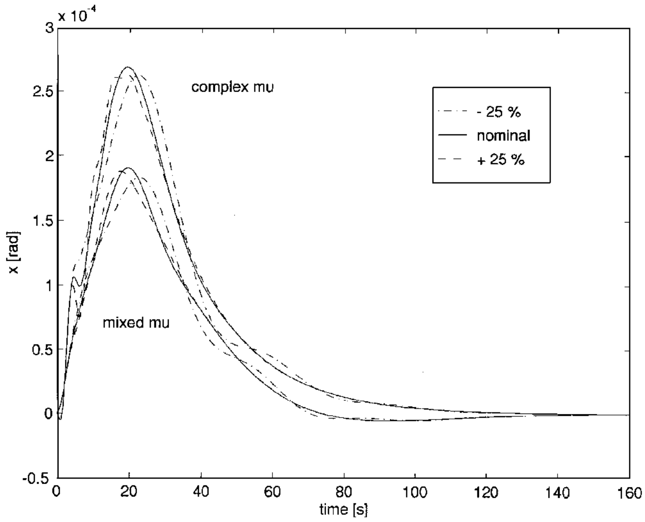

Fig. 11 Yaw angle displacement for full-order complex μ and fixed fifth-order mixed μ designs, ±25% variation in structural frequency.

Figure 11 shows the yaw angle displacement for a 1.0-in.-lbf step disturbance torque for the full-order complex μ design and the fixed fifth-order mixed μ design. Both designs satisfy specifications 1 and 2. The mixed μ controller, however, reduces the amplitude of the displacement and achieves a slightly shorter settling time indicating improved robust performance.

The results of the flexible satellite example confirm the findings on the benchmark problem. The mixed μ controllers reduce conservatism in robust control design but tend to be of relatively large order. The fixed seventh-order controller synthesized with the new method is able to improve over a corresponding reduced-order design. Moreover, a fixed fifth-order controller could be designed whereas a reduced fifth-order controller was not internally stabilizing.

In an extension to the results presented in this paper, a comprehensive robust control study has been carried out for a hypersonic vehicle in Ref. 30. Flight controllers as low as fifth-order have been designed for a 47th-order generalized plant with negligible sacrifices in robust performance.

Conclusions

A new method to synthesize fixed-order robust controllers for mixed real/complex uncertainties has been presented. The benefit of the new method is its ability to constrain the order of the controller a priori in the design process. Applications to a two-mass/spring benchmark problem and a flexible satellite example confirm the usefulness of the method. In both cases, mixed μ controllers are shown to improve over the achievable robust performance levels of complex μ designs. However, these controllers tend to be of relatively large order. The fixed-order mixed μ controllers synthesized with the presented method are of considerably lower order and closely approximate the robust performance measures of the full-order designs. They are in general superior to corresponding reduced-order designs.

Acknowledgments

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